

Chapter 3

A Compact Macromodel of World Economic and Demographic Growth

Before proposing this model, it appears necessary to consider in more detail the model developed by Michael Kremer (1993).

Kremer assumes that overall output produced by the world economy equals

$$G = rTN^\alpha V^{1-\alpha},$$

where G is output, T is the level of technology, N is population, V is land, r and α ($0 < \alpha < 1$) are parameters. Actually Kremer uses a variant of the Cobb-Douglas production function. Kremer further qualifies that variable V is normalized to one. The resultant equation for output is:

$$G = rTN^\alpha, \quad (3.1)$$

where r and α are constants.

Further Kremer uses the Malthusian assumption, formulating it in the following way: "In this simplified model I assume that population adjusts instantaneously to \bar{N} " (Kremer 1993: 685). Value \bar{N} in this model corresponds to population size, at which it produces equilibrium level of per capita income \bar{g} , whereas "population increases above some steady state equilibrium level of per capita income, \bar{g} , and decreases below it" (Kremer 1993: 685).

Thus, the equilibrium level of population \bar{N} is

$$\bar{N} = \left(\frac{\bar{g}}{rT} \right)^{\frac{1}{\alpha-1}}. \quad (3.2)$$

Hence, the equation for population size is not actually dynamic. In Kremer's model the dynamic element is introduced by a supplementary equation for technological growth. Kremer uses the following assumption of the Endogenous Technological Growth theory, which we have already used above for the development of the first compact macromodel (Kuznets 1960; Grossman and

Helpman 1991; Aghion and Howitt 1992, 1998; Simon 1977, 1981, 2000; Komlos and Nefedov 2002; Jones 1995, 2003, 2005 *etc.*):

"High population spurs technological change because it increases the number of potential inventors...¹. All else equal, each person's chance of inventing something is independent of population. Thus, in a larger population there will be proportionally more people lucky or smart enough to come up with new ideas" (Kremer 1993: 685); thus, "the growth rate of technology is proportional to total population" (Kremer 1993: 682). (3.3)

Since this supposition was first proposed by Simon Kuznets (1960), we shall denote the respective type of dynamics as "Kuznetsian"; and we shall denote as "Malthusian-Kuznetsian" those systems where "Kuznetsian" population-technological dynamics is combined with "Malthusian" demographics.

The Kuznetsian assumption is expressed mathematically by Kremer in the following way:

$$\frac{dT}{dt} : T = bN, \quad (3.4)$$

where b is average innovating productivity per person.

Note that this implies that the dynamics of absolute technological growth rate can be described by the following equation:

$$\frac{dT}{dt} = bNT. \quad (3.5)$$

Kremer

"Since population is limited by technology, the growth rate of population is proportional to the growth rate of technology. Since the growth rate of technology is proportional to the level of population, the growth rate of population must also be proportional to the level of population. To see this formally, take the logarithm of the population determination equation, [(3.2)], and differentiate with respect to time:

$$\frac{dN}{dt} : N = \frac{1}{1 - \alpha} \left(\frac{dT}{dt} : T \right).$$

Substitute in the expression for the growth rate of technology from [(3.4)], to obtain

¹ "This implication flows naturally from the nonrivalry of technology... The cost of inventing a new technology is independent of the number of people who use it. Thus, holding constant the share of resources devoted to research, an increase in population leads to an increase in technological change" (Kremer 1993: 681).

$$\frac{dN}{dt} : N = \frac{b}{1-\alpha} N^2 \quad (\text{Kremer 1993: 686}). \quad (3.6)$$

Note that multiplying both parts of equation (3.6) by N we get

$$\frac{dN}{dt} = aN^2, \quad (2.4')$$

where a equals

$$a = \frac{b}{1-\alpha}.$$

Of course, the same equation can be also written as

$$\frac{dN}{dt} = \frac{N^2}{C}, \quad (2.4)$$

where C equals

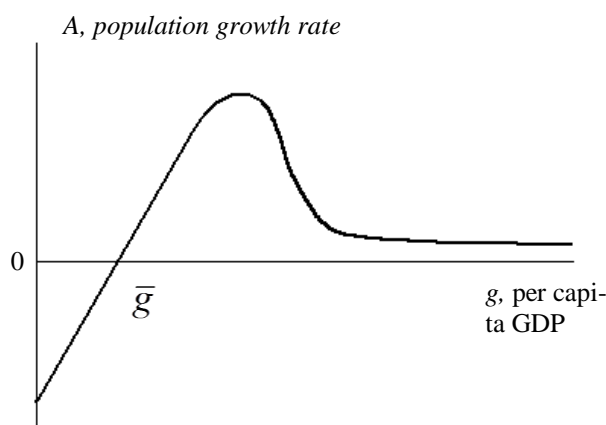
$$C = \frac{1-\alpha}{b}.$$

Thus, Kremer's model produces precisely the same dynamics as the ones of von Foerster and Kapitza (and, consequently, it has just the same phenomenal fit with the observed data). However, it also provides a very convincing explanation WHY throughout most of the human history the absolute world population growth rate tended to be proportional to N^2 . Within both models the growth of population from, say, 10 million to 100 million will result in the growth of dN/dt 100 times. However, von Foerster and Kapitza failed to explain convincingly why dN/dt tended to be proportional to N^2 . Kremer's model explains this in what seems to us a rather convincing way (though Kremer himself does not appear to have spelled this out in a sufficiently clear way). The point is that the growth of the world population from 10 to 100 million implies that the human technology also grew approximately 10 times (as it turns out to be able to support a ten times larger population). On the other hand, the growth of population 10 times also implies 10-fold growth of the number of potential inventors, and, hence, 10-fold increase in the relative technological growth rate. Hence, the absolute technological growth will grow $10 \times 10 = 100$ times (in accordance to equation (3.5)). And as N tends to the technologically determined carrying capacity ceiling, we have all grounds to expect that dN/dt will also grow just 100 times.

Though Kremer's model provides a virtual explanation of how the World System's techno-economic development, in connection with demographic dynamics, could lead to hyperbolic population growth, Kremer did not specify his model to such an extent that it could also describe the economic development of the World System and that such a description could be tested empirically.² Nevertheless, it appears possible to propose a very simple mathematical model describing both the demographic and economic development of the World System up to 1973 using the same assumptions as those employed by Kremer.

Kremer's analysis suggests the following relationship between per capita GDP and population growth rate (see Diagram 3.1):

Diagram 3.1. Relationship between per capita GDP and Population Growth Rate according to Kremer (1993)



This suggests that in the lower range of per capita GDP the influence of this variable on the dynamics of population growth can be described with the following equation:

$$\frac{dN}{dt} = aSN, \tag{3.7}$$

where S is surplus, which is produced per person over the amount (m), which is minimally necessary to reproduce the population with a zero growth rate in a Malthusian system (thus, $S = g - m$, where g denotes per capita GDP).

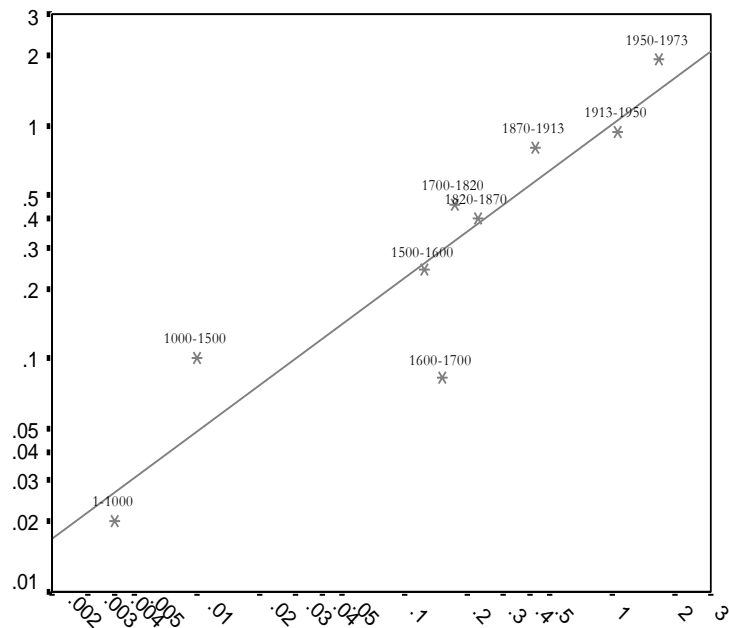
² In fact, such an operationalization did not make sense at the time when Kremer's article was submitted for publication, as the long term empirical data on the world GDP dynamics were not simply available at that moment.

Note that this model generates predictions that can be tested empirically. For example, the model predicts that relative world population growth ($r_N = \frac{dN}{dt} : N$) should be linearly proportional to the world per capita surplus production:

$$r_N = aS . \quad (3.8)$$

The empirical test of this hypothesis has supported it. The respective correlation has turned out to be in the predicted direction, very strong ($R = 0.961$), and significant beyond any doubt ($p = 0.00004$) (see Diagram 3.2):

Diagram 3.2. Correlation between the per capita surplus production and world population growth rates for 1–1973 CE (scatterplot with fitted regression line)



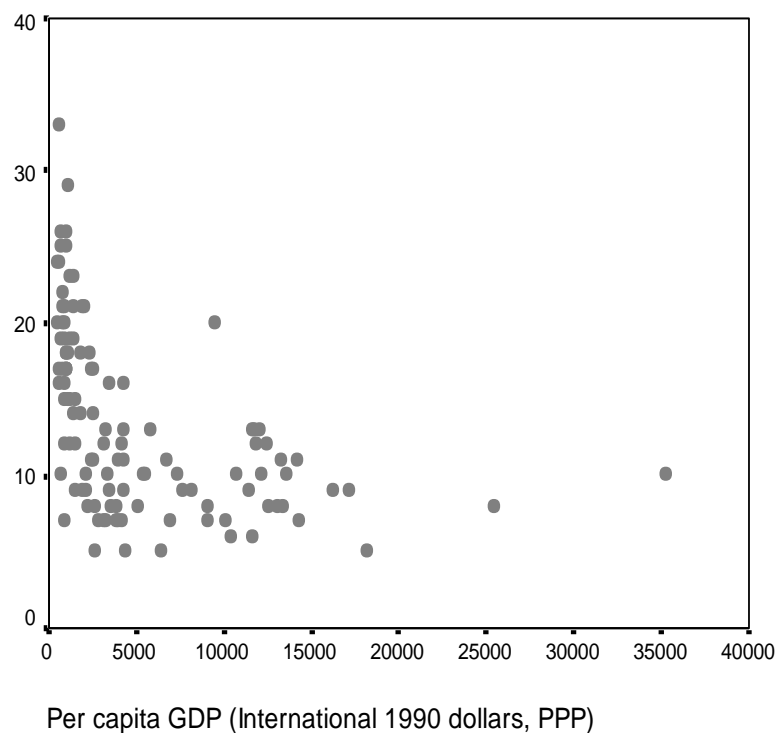
Per capita surplus production (thousands of 1990 int.dollars,PPP)

NOTES: $R = 0.961$, $p = 0.00004$. Data source – Maddison 2001; Maddison's estimate of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev (1996: 55–97). S values were calculated on the basis of m estimated as 440 international 1990 dollars in purchasing power

parity (PPP); for the justification of this estimate see Korotayev, Malkov, and Khaltourina 2005: 43–51.

The mechanisms of this relationship are perfectly evident. In the range \$440–3500³ the per capita GDP growth leads to very substantial improvements in nutrition, health care, sanitation *etc.* resulting in a precipitous decline of death rates (see, *e.g.*, Diagram 3.3):

Diagram 3.3. Correlation between per capita GDP and death rate for countries of the world in 1975

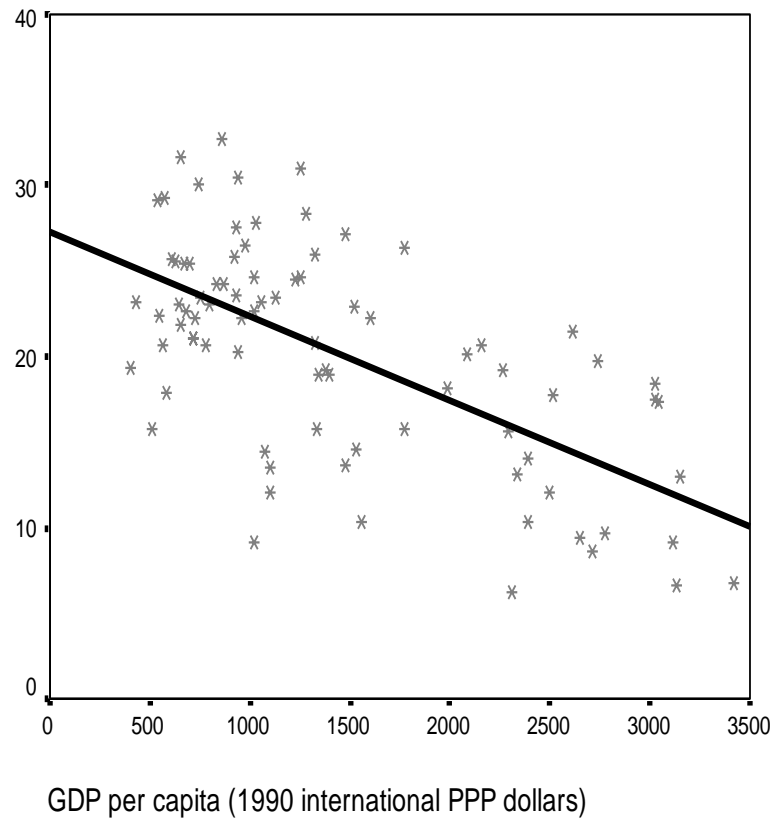


NOTE: data sources – Maddison 2001 (for per capita GDP), World Bank 2005 (for death rate).

For example, for 1960 the correlation between per capita GDP and death rate for \$440–3500 range reaches -0.634 ($p = 0.000000001$) (see Diagram 3.4):

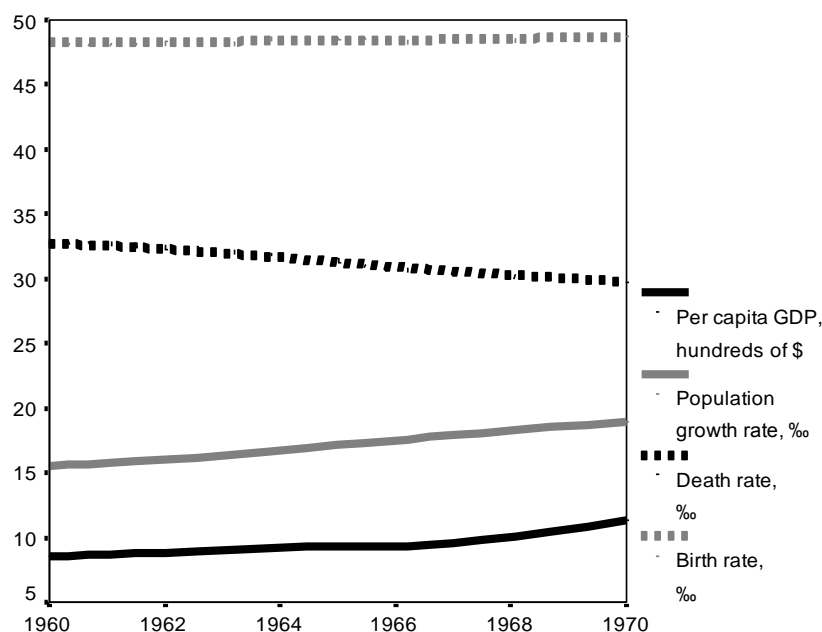
³ Here and throughout the GDP is measured in 1990 international purchasing power parity dollars after Maddison (2001) if not stated otherwise.

Diagram 3.4. Correlation between per capita GDP and death rate for countries of the world in 1960 (for \$440–3500 range)



NOTE: $R = -0.634$, $p = 0.0000000001$. Data sources – Maddison 2001 (for per capita GDP), World Bank 2005 (for death rate).

Note that during the earliest stages of demographic transition (corresponding just to the range in question) the decline of the death rates is not accompanied by a corresponding decline of the birth rates (*e.g.*, Chesnais 1992); in fact, they can even grow (see, *e.g.*, Diagram 3.5).

Diagram 3.5. Economic and demographic dynamics in Sierra Leone, 1960–1970

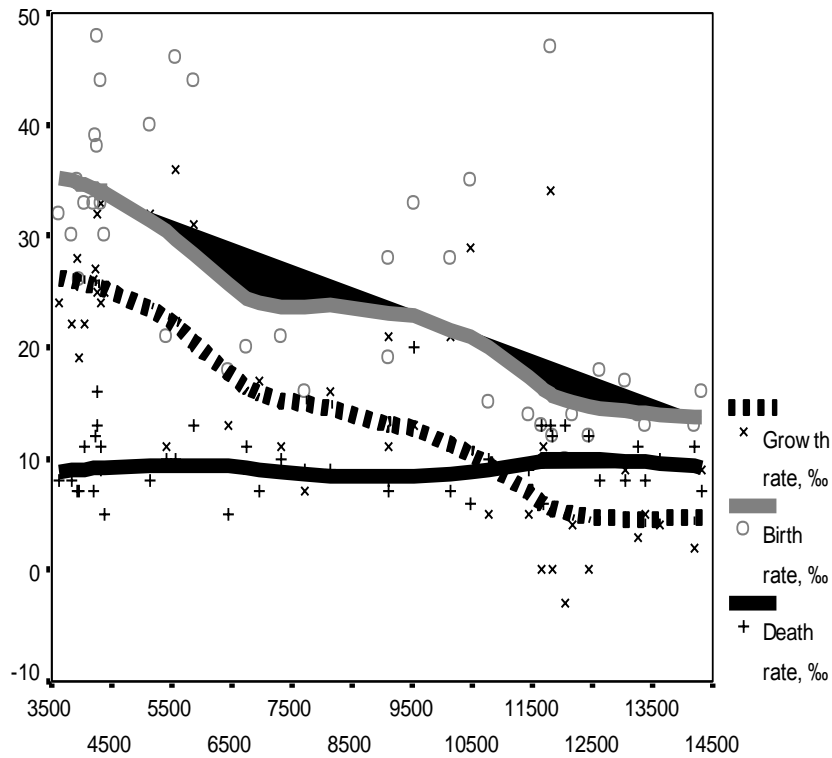
NOTE: Data sources – Maddison 2001 (for per capita GDP), World Bank 2005 (for demographic variables).

Indeed, say a decline of the death rates from, *e.g.*, 50 to 25‰ implies the growth of life expectancies from 20 to 40 years, whereas a woman living 40 years can bear many more babies than when she dies at the age of 20. In many countries the growth of the fertility rates in the range in question was also connected with the reduction of spacing between births, due to such modernization-produced changes as an abandonment of traditional postpartum sex taboos, or due to a reduction of lactation periods (given the growing availability of foods that can serve as substitutes for maternal milk). Against this background the radical decline of the death rates at the earliest stages of demographic transition is accompanied by as a radical increase of the population growth rates.

Note that equation (3.7) is not valid for the GDP per capita range > \$3500. In this "post-Malthusian" range the death rate reaches bottom levels and then starts slightly growing due to the population aging, whereas the growth of such variables as education (and especially female education), level of social security subsystem development, growing availability of more and more sophisticated family planning techniques *etc.* leads to a sharp decline of birth rates (see, *e.g.*, Hollingsworth 1996, Bongaarts 2003, Korotayev, Malkov, and Khaltouri-

na 2005). As a result in this range the further growth of per capita GDP leads not to the increase of the population growth rates, but to their substantial decrease (see Diagram 3.6):

Diagram 3.6. Relationships between per capita GDP (1990 international PPP dollars, X-axis), Death Rates (%o, Y-axis), Birth Rates (%o, Y-axis), and Population Growth Rates⁴ (%o, Y-axis), nations with per capita GDP > \$3500, 1975, scatterplot with fitted Lowess lines

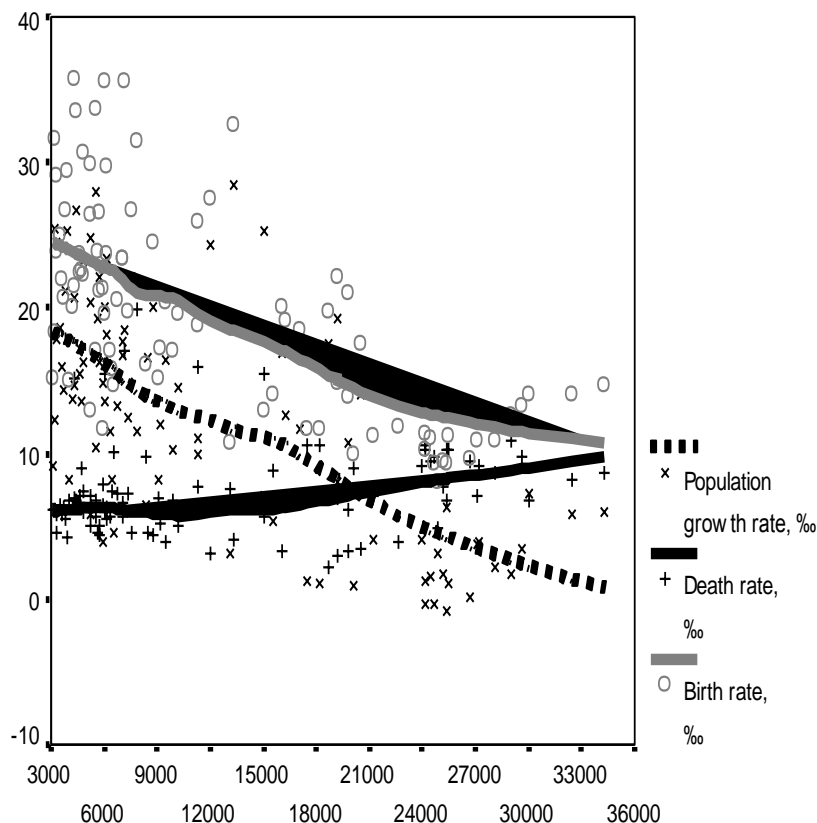


NOTE: Data sources – Maddison 2001 (for per capita GDP), World Bank 2005 (for the other data).

⁴ Internal ("natural") population growth rate calculated as birth rate minus death rate. We used this variable instead of standard growth rate, as the latter takes into account the influence of emigration and immigration processes, which notwithstanding all their importance are not relevant for the subject of this chapter, because though they could affect in a most significant way the population growth rates of particular countries, they do not affect the world population growth.

This pattern is even more pronounced for the world countries of 2001, as between 1975 and 2001 the number of countries that had moved out of the first phase of demographic transition to the "post-Malthusian world" substantially increased (see Diagram 3.7):

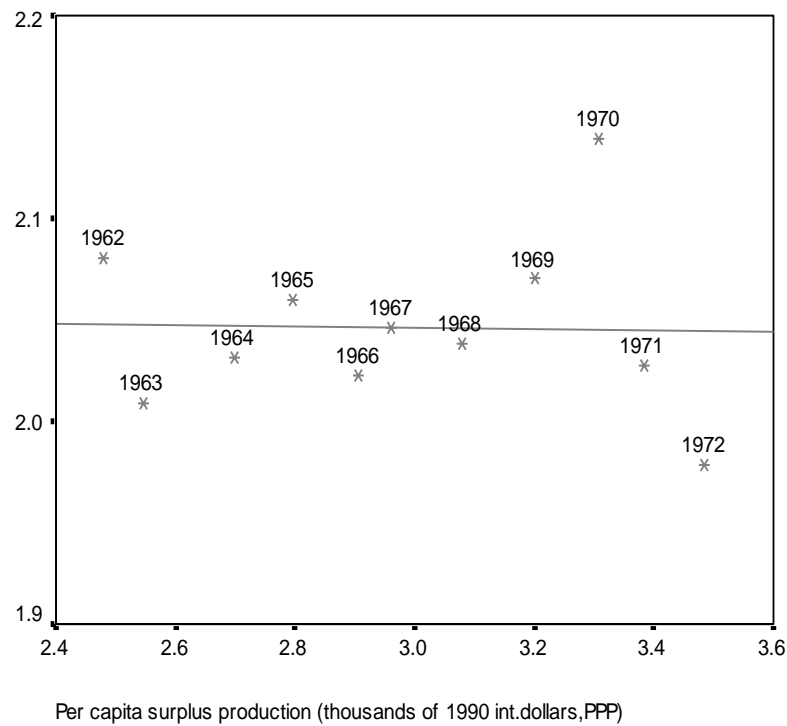
Diagram 3.7. Relationships between per capita GDP (2001 PPP USD, X-axis), Death Rates (‰, Y-axis), Birth Rates (‰, Y-axis), and Population Growth Rates (‰, Y-axis), nations with per capita GDP > \$3000, 2001, scatterplot with fitted Lowess lines



NOTE: Data source – World Bank 2005. Omitting the former Communist countries of Europe, which are characterized by a very specific pattern of demographic growth (or, in fact, to be more exact, demographic decline – see, *e.g.*, Korotayev, Malkov, and Khaltourina 2005: 302–29).

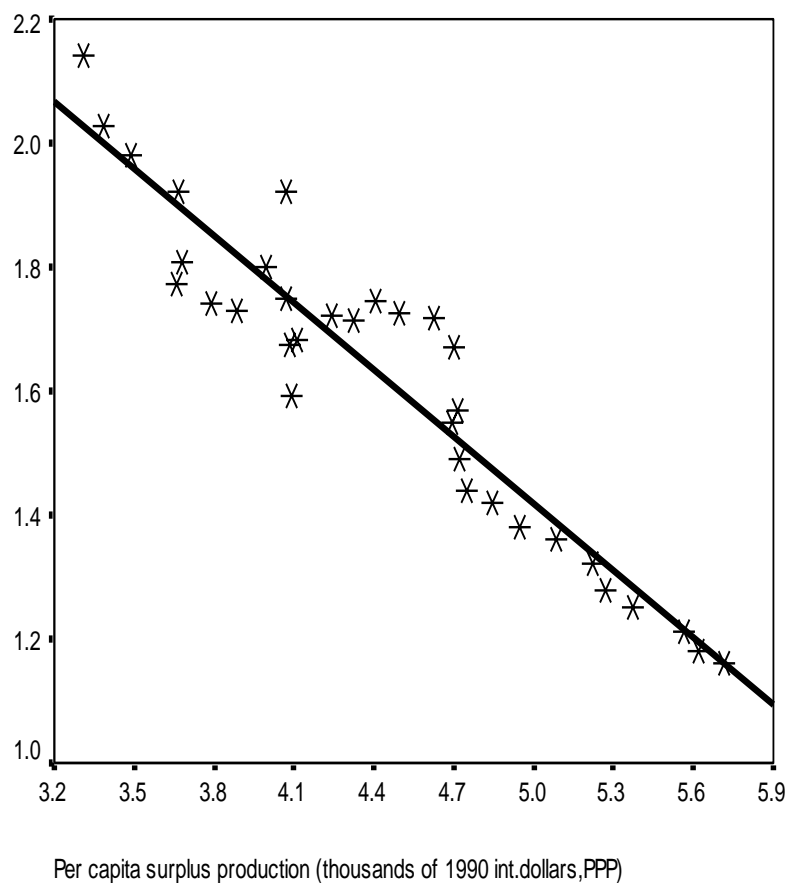
It is also highly remarkable that as soon as the world per capita GDP (g) approached \$3000 and exceeded this (thus, with $S \sim \$2500$), the positive correlation between S and r first dropped to zero (see Diagram 3.8); beyond $S = 3300$ ($g \sim 3700$) it becomes strongly negative (see Diagram 3.9), whereas in the range of $S > 4800$ ($g > 5200$) this negative correlation becomes almost perfect (see Diagram 3.10):

Diagram. 3.8. Correlation between the per capita surplus production and world population growth rates for the range $\$2400 < S < \3500 (scatterplot with fitted regression line)



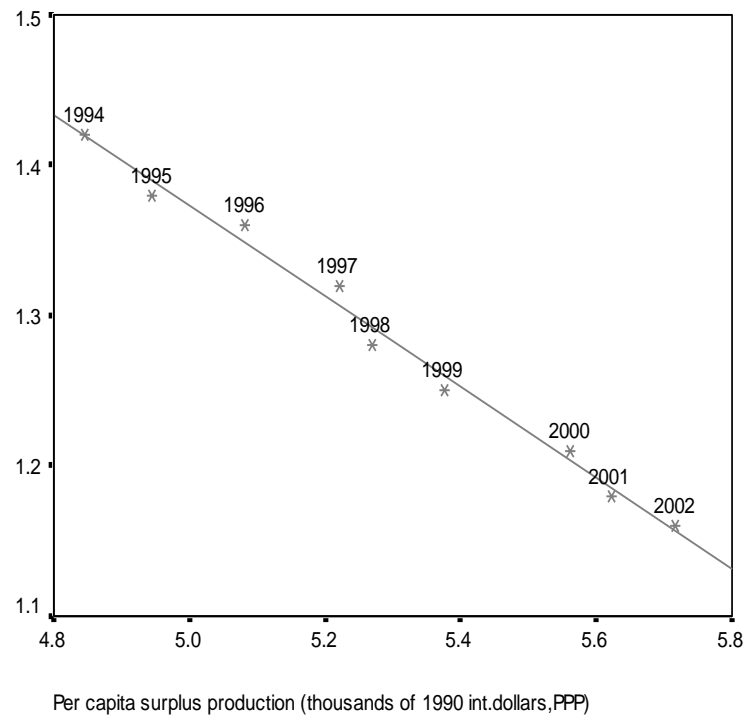
NOTES: $R = -0.028$, $p = 0.936$. Data source – Maddison 2001.

Diagram 3.9. Correlation between the per capita surplus production and world population growth rates for $S > \$4800$ (scatterplot with fitted regression line)



NOTES: $R = -0.946$, $p = 0.0000000000000003$. Data sources – Maddison 2001 (for the world per capita GDP, 1970–1998 and the world population growth rates, 1970–1988); World Bank 2005 (for the world per capita GDP, 1999–2002); United Nations 2005 (for the world population growth rates, 1989–2002).

Diagram 3.10. Correlation between the per capita surplus production and world population growth rates for $S > \$4800$ (scatterplot with fitted regression line)



NOTES: $R = -0.995$, $p = 0.00000004$. Data sources – Maddison 2001 (for the world per capita GDP, 1994–1998); World Bank 2005 (for the world per capita GDP, 1999–2002); United Nations 2005 (for the world population growth rates).

Thus, to describe the relationships between economic and demographic growth in the "post-Malthusian" GDP per capita range of $> \$3000$, equation (3.7) should be modified, which is quite possible, but which goes beyond the scope of this chapter aimed at the description of economic and demographic macrodynamics of the "Malthusian" period of human history; this and will be done in the subsequent chapters.

As was already noted by Kremer (1993: 694), in conjunction with equation (3.1) an equation of type (3.7) "in the absence of technological change [that is if $T = \text{const}$] reduces to a purely Malthusian system, and produces behavior similar to the logistic curve biologists use to describe animal populations facing fixed resources" (actually, we would add to Kremer's biologists those social scientists who model pre-industrial demographic cycles – see, *e.g.*, Usher 1989;

Chu and Lee 1994; Nefedov 2002, 2004; Malkov 2002, 2003, 2004; Malkov and Sergeev 2002, 2004; Malkov, Selunskaja, and Sergeev 2005; Turchin 2003; Korotayev, Malkov, and Khaltourina 2005).

Note that with a constant relative technological growth rate ($\frac{\dot{T}}{T} = r_T = \text{const}$) within this model (combining equations (3.1) and (3.7)) we will have both constant relative population growth rate ($\frac{\dot{N}}{N} = r_N = \text{const}$, and thus the population will grow exponentially) and constant S . Note also that the higher value of r_T we take, the higher value of constant S we get.

Let us show this formally.

Take the following system:

$$G = gTN^\alpha \tag{3.1}$$

$$\frac{dN}{dt} = aSN \tag{3.7}$$

$$\frac{dT}{dt} = cT, \tag{3.9}$$

where $S = \frac{G}{N} - m$.

Equation (3.9) evidently gives $T = T_0 e^{ct}$. Thus, $G = gT_0 e^{ct} N^\alpha$, and consequently

$$\frac{dN}{dt} = a \left(\frac{gT_0 e^{ct} N^\alpha}{N} - m \right) N = agT_0 e^{ct} N^\alpha - amN .$$

This is known as a Bernoulli differential equation: $\frac{dy}{dx} = f(x)y + g(x)y^\alpha$,

which has the following solution:

$$y^{1-\alpha} = Ce^{F(x)} + (1-\alpha)e^{F(x)} \int e^{-F(x)} g(x) dx ,$$

where $F(x) = (1-\alpha) \int f(x) dx$, and C is constant.

In the case considered above, we have

$$N^{1-\alpha} = Ce^{F(t)} + (1-\alpha)e^{F(t)} \int e^{-F(t)} agT_0 e^{ct} dt,$$

where $F(t) = (1-\alpha) \int (-am) dt = (\alpha-1)amt$.

$$\text{So } N^{1-\alpha} = Ce^{(\alpha-1)amt} + (1-\alpha)agT_0 e^{(\alpha-1)amt} \int e^{-(\alpha-1)amt} e^{ct} dt$$

$$N^{1-\alpha} = e^{-(1-\alpha)amt} \left(C + (1-\alpha)agT_0 \int e^{(c+(1-\alpha)am)t} dt \right)$$

$$N^{1-\alpha} = e^{-(1-\alpha)amt} \left(C + \frac{(1-\alpha)agT_0}{c + (1-\alpha)am} e^{(c+(1-\alpha)am)t} \right)$$

This result causes the following equation for S :

$$S = \frac{gT_0 e^{ct} N^\alpha}{N} - m = gT_0 e^{ct} N^{\alpha-1} - m = \frac{gT_0 e^{(c+(1-\alpha)am)t}}{C + \frac{(1-\alpha)agT_0}{c + (1-\alpha)am} e^{(c+(1-\alpha)am)t}} - m;$$

$$S = \frac{I}{\frac{C}{gT_0} e^{-(c+(1-\alpha)am)t} + \frac{(1-\alpha)a}{c + (1-\alpha)am}} - m.$$

Since $c > 0$ and $(1-\alpha) > 0$, it is clear that $c + (1-\alpha)am > 0$.

Consequently $e^{-(c+(1-\alpha)am)t} \rightarrow 0$ as $t \rightarrow \infty$.

This means that $S \xrightarrow{t \rightarrow \infty} \frac{c + (1-\alpha)am}{(1-\alpha)a} - m$, or finally

$$S \rightarrow \frac{c}{(1-\alpha)a}, \text{ as } t \rightarrow \infty.$$

Note that coefficient c in equation (3.9) is nothing else but just the relative technological growth rate ($r_T = dT/dt : T$). Thus within the system (3.1)-(3.7)-(3.9), with a constant relative technological growth rate ($r_T = c$), the per capita surplus (S) would also tend to some constant, and the higher value of relative technological growth rate ($c = r_T$) we take, the higher value of constant S we get at the end.

This, of course, suggests that in the growing "Malthusian" systems S could be regarded as a rather sensitive indicator of the speed of technological growth. Indeed, within Malthusian systems in the absence of technological growth the demographic growth will lead to S tending to 0, whereas a long-term systematic production of S will be only possible with systematic technological growth.⁵

⁵ It might make sense to stress that it is not coincidental that we are speaking here just about the long-term perspective, as in a shorter-term perspective it would be necessary to take into account that within the actual Malthusian systems S was also produced quite regularly at the recovery

Now replace $\frac{\dot{T}}{T} = r_T = \text{const}$ with Kremer's technological growth equation (3.5) and analyze the resultant model:

$$G = aTN^\alpha, \quad (3.1)$$

$$\frac{dN}{dt} = bSN, \quad (3.7)$$

$$\frac{dT}{dt} = cNT. \quad (3.5)$$

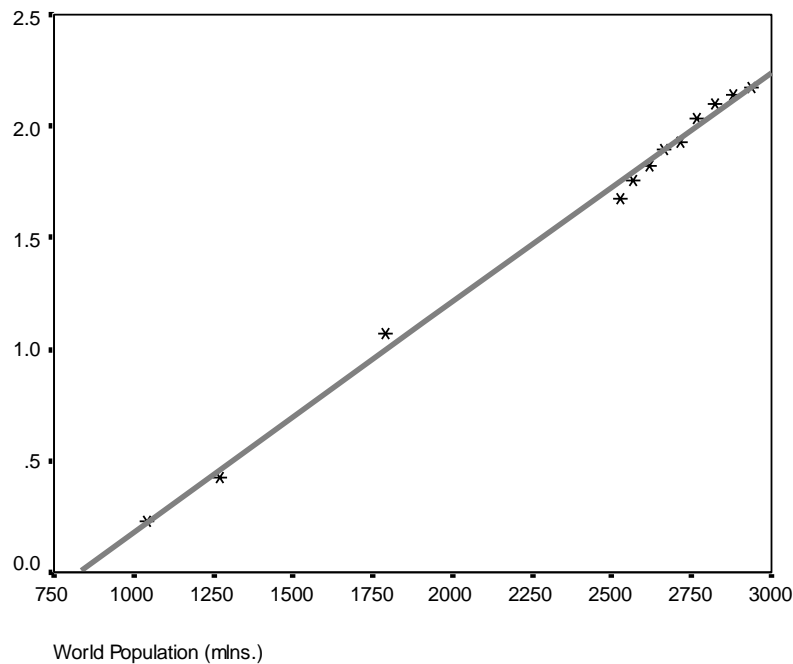
Within this model, quite predictably, S can be approximated as kr_T . On the other hand, within this model, by definition, r_T is directly proportional to N . Thus, the model generates an altogether not so self-evident (one could say even a bit unlikely) prediction – that throughout the "Malthusian-Kuznetsian" part of the human history the world per capita surplus production must have tended to be directly proportional to the world population size. This hypothesis, of course, deserves to be empirically tested. In fact, our tests have supported it.

Our test for the whole part of human history for which we have empirical estimates for both the world population and the world GDP (that is for 1–2002 CE⁶) has produced the following results: $R^2 = 0.98$, $p < 10^{-16}$, whereas for the period with the most pronounced "Malthusian-Kuznetsian" dynamics (1820–1958) the positive correlation between the two variables is almost perfect (see Diagram 3.11):

phases of pre-Industrial political-demographic cycles (following political-demographic collapses as a result of which the surviving population found itself abundantly provided with resources). However, after the recovery phases the continuing production of significant amounts of S (and, hence, the continuing significant population growth) was only possible against the background of significant technological growth (see, *e.g.*, Korotayev, Malkov, Khaltourina 2005: 160–228). Note also that S produced at the initial (recovery) phases of political-demographic cycles in no way can explain the millennial trend towards the growth of S which was observed for many centuries before most of the world population moved to the second phase of demographic transition (*e.g.*, Maddison 2001) and that appears to have been produced just by the accelerating technological growth.

⁶ Data sources – Maddison 2001 (world population and GDP, 1–1998 CE), World Bank 2005 (world GDP, 1999–2002), United Nations 2005 (world population, 1999–2002).

Diagram 3.11. Correlation between world population and per capita surplus production (1820–1958)



NOTE: $R^2 > 0.996$, $p < 10^{-12}$.

Note that as within a Malthusian-Kuznetsian system S can be approximated as kN , equation (3.7) may be approximated as $dN/dt \sim k_1 N^2$ (2.4), or, of course, as $dN/dt \sim N^2 / C$ (2.4); thus, Kapitzka's equation (2.4) turns out to be a by-product of the model under consideration.

Thus, we arrive at the following:

$$S \sim k_1 r_T,$$

$$r_T = k_2 N.$$

Hence,

$$dS/dt \sim k r_T/dt = k_3 dN/dt.$$

This implies that for the "Malthusian-Kuznetsian" part of human history dS/dt can be approximated as $k_4 dN/dt$.

On the other hand, as dN/dt in the original model equals aSN , this, of course, suggests that for the respective part of the human history both the eco-

conomic and demographic World System dynamics may be approximated by the following unlikely simple mathematical model:⁷

$$\frac{dN}{dt} = aSN, \quad (3.7)$$

$$\frac{dS}{dt} = bNS, \quad (3.10)$$

where N is the world population, and S is surplus, which is produced per person with the given level of technology over the amount, which is minimally necessary to reproduce the population with a zero growth rate.

The world GDP is computed using the following equation:

$$G = mN + SN, \quad (3.11)$$

where m denotes the amount of per capita GDP, which is minimally necessary to reproduce the population with a zero growth rate, and S denotes "surplus" produced per capita over m at the given level of the world-system technological development.

Note that this model does not contain any variables for which we do not have empirical data (at least for 1–1973) and, thus, a full empirical test for this model turns out to be perfectly possible.⁸

Incidentally, this model implies that the absolute rate of the world population growth (dN/dt) should have been roughly proportional to the absolute rate of the increase in the world per capita surplus production (dS/dt), and, thus (assuming the value of necessary product to be constant), to the absolute rate of the world per capita GDP growth, with which dS/dt will be measured thereafter. Note that among other things this could help us to determine the proportion between coefficients a and b .

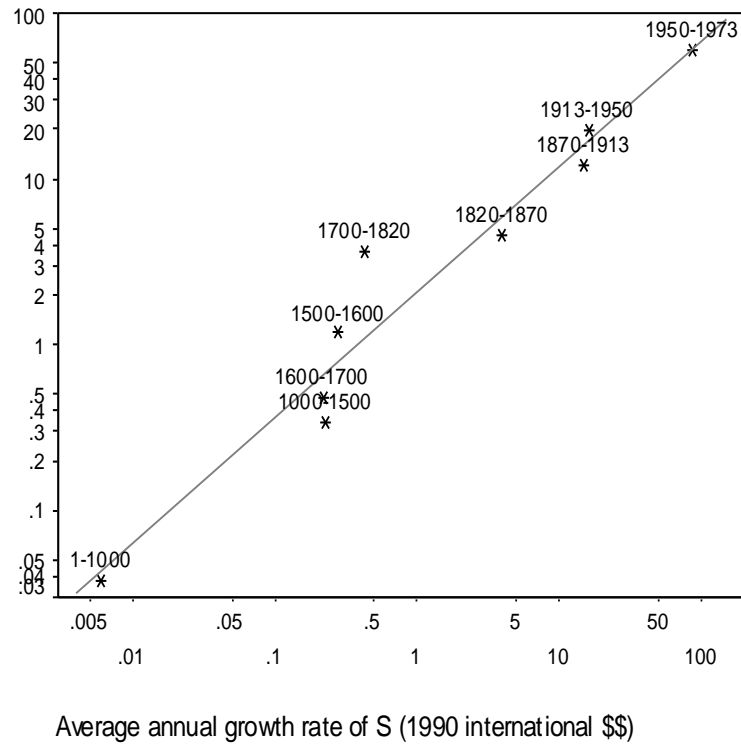
Thus, if the model suggested by us has some correspondence to reality, one has grounds to expect that in the "Malthusian-Kuznetsian" period of the human history the absolute world population growth rate (dN/dt) was directly proportional to the absolute growth rate of the world per capita surplus production

⁷ Note that this model only describes the Malthusian-Kuznetsian World System in a dynamically balanced state (when the observed world population is in a balanced correspondence with the observed technological level). To describe the situations with N disproportionately low or high for the given level of technology (and, hence, disproportionately high or low S) one would need, of course, the unapproximated version of the model ((3.1) – (3.7) – (3.5)). Note, that in such cases N will either grow, or decline up to the dynamic equilibrium level, after which the developmental trajectory will follow the line described by the (3.7) – (3.10) model.

⁸ This refers particularly to the long-range data on the level of world technological development (T), which do not appear to be available now.

(dS/dt). The correlation between these two variables looks as follows (see Diagram 3.12):

Diagram 3.12. Correlation between World Average Annual Absolute Growth Rate of Per Capita Surplus Production (S , 1990 PPP international dollars) and Average Annual Absolute World Population (N) Growth Rate (1 – 1973 CE), scatterplot in logarithmic scale with a regression line



NOTE: Data source – Maddison 2001; Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev (1996: 55–97).

Regression analysis of this dataset has given the following results (see Table 3.1):

Table 3.1. Correlation between World Average Annual Absolute Growth Rate of Per Capita Surplus Production (S , 1990 PPP international dollars) and Average Annual Absolute World Population (N) Growth Rate (1 – 1950 CE), (regression analysis)

Model	Unstandardized Coefficients		Standardized Coefficients	t	Sig.
	B	Std. Error	Beta		
(Constant)	0.820	0.935		0.876	0.414
World Average Annual Absolute Growth Rate of Per Capita Surplus Production (1990 PPP international dollars / year)	0.981	0.118	0.959	8.315	<0.001
Dependent variable: Average Annual Absolute World Population Growth Rate (mlns. / year)					

$$R = 0.96, R^2 = 0.92$$

Note that the constant in this case is very small within the data scale, statistically insignificant, and lies within the standard error from 0, which makes it possible to equate it with 0. In this case regression analysis gives the following results (see Table 3.2):

Table 3.2. Correlation between World Average Annual Absolute Growth Rate of Per Capita Surplus Production (S , 1990 PPP international dollars) and Average Annual Absolute World Population (N) Growth Rate (1 – 1950 CE), (regression analysis, not including constant in equation)

Model	Unstandardized Coefficients		Standardized Coefficients	t	Sig.
	B	Std. Error	Beta		
World Average Annual Absolute Growth Rate of Per Capita Surplus Production (1990 PPP international dollars / year)	1.04	0.095	0.972	10.94	<0.001
Dependent variable: Average Annual Absolute World Population Growth Rate (mlns. / year)					

$$R = 0.97, R^2 = 0.945$$

Thus, just as implied by our second model, in the "Malthusian" period of human history we do observe a strong linear relationship between the annual absolute world population growth rates (dN/dt) and the annual absolute growth rates of per capita surplus production (dS/dt). This relationship can be described mathematically with the following equation:

$$\frac{dN}{dt} = 1.04 \frac{dS}{dt} ,$$

where N is the world population (in millions), and S is surplus (in 1990 PPP international dollars), which is produced per person with the given level of technology over the amount, which is minimally necessary to reproduce the population with a zero growth rate.

Note that according to model (3.7, 3.10),

$$\frac{dN}{dt} = \frac{a}{b} \frac{dS}{dt} .$$

Thus, it becomes possible to express coefficient b through coefficient a :

$$\frac{a}{b} = 1.04 ,$$

consequently:

$$b = \frac{a}{1.04} = 0.96a .$$

As a result, for the period under consideration it appears possible to simplify the second compact macromodel, leaving in it just one free coefficient:

$$\frac{dN}{dt} = aSN , \quad (3.7)$$

$$\frac{dS}{dt} = 0.96aNS , \quad (3.12)$$

With our two-equation model we start the simulation in the year 1 CE and do annual iterations with difference equations derived from the differential ones:

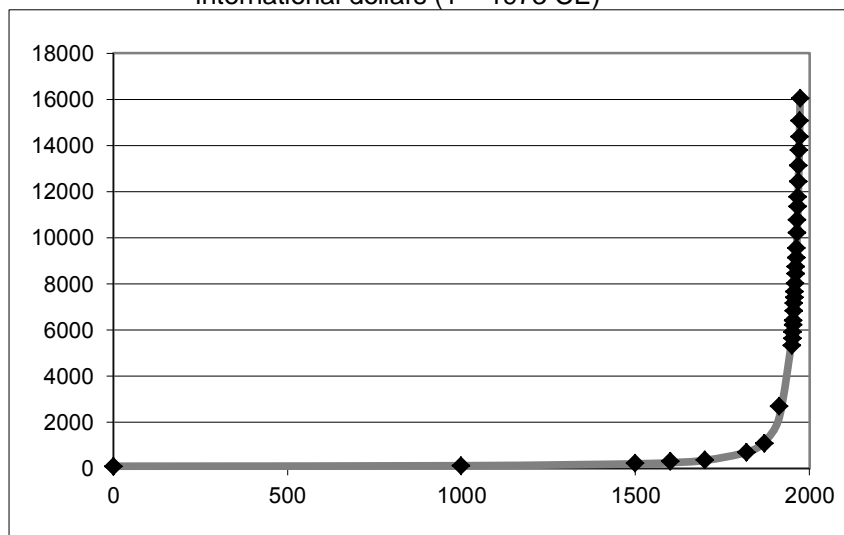
$$\begin{aligned} N_{i+1} &= N_i + aS_iN_i, \\ S_{i+1} &= S_i + 0.96aN_iS_i. \end{aligned}$$

The world GDP is calculated using equation (3.11).

We choose the following values of the constants and initial conditions in accordance with historical estimates of Maddison (2001): $N_0 = 230.82$ (in millions); $a = 0.000011383$; $S_0 = 4.225$ (in International 1990 PPP dollars).⁹

The outcome of the simulation, presented in Diagram 3.13 indicates that the compact macromodel in question is actually capable of replicating quite reasonably the world GDP estimates of Maddison (2001):

Diagram 3.13. Predicted and Observed Dynamics of the World GDP Growth, in billions of 1990 PPP international dollars (1 – 1973 CE)



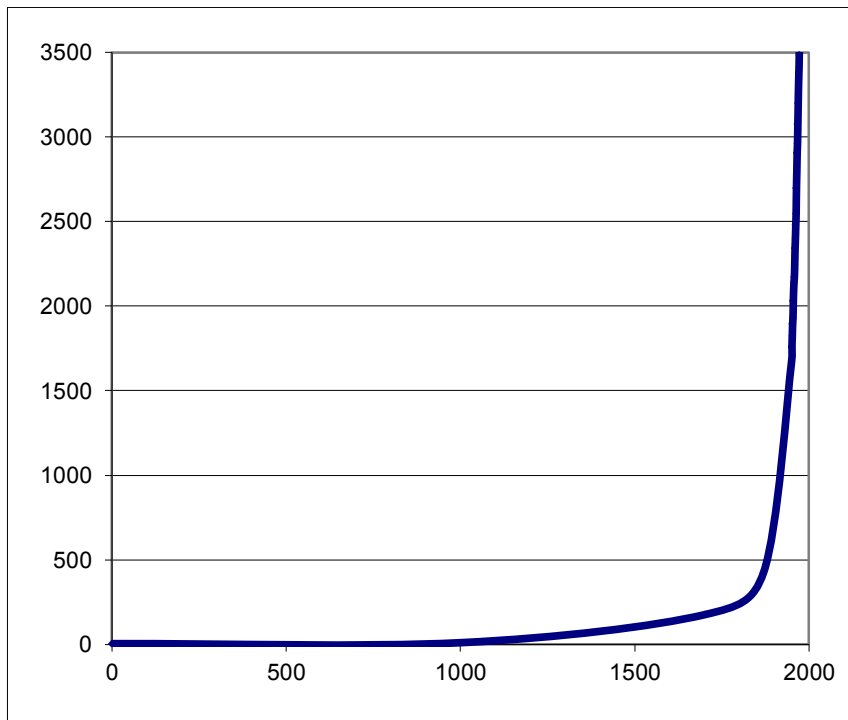
NOTE: The solid grey curve has been generated by the model; black markers correspond to the estimates of world GDP by Maddison (2001).

⁹ The value of S_0 was calculated with equation $S = G/N - m$ on the basis of Maddison's (2001) estimates for the year 1 CE. He estimates the world population in this year as 230.82 million, the world GDP as \$102.536 billion (in 1990 PPP international dollars), and hence, the world per capita GDP production as \$444.225. Maddison estimates the subsistence level per capita annual GDP production as \$400 (2001: 260, 264). However, already by 1 CE most population of the world lived in rather complex societies, where the population reproduction even at zero level still required considerable production over subsistence level to maintain various infrastructures (transportation, legal, security, administrative and other subsystems *etc.*), without which even the simple reproduction of complex societies is impossible almost by definition. Note that the fall of per capita production in complex agrarian societies to subsistence level tended to lead to state breakdowns and demographic collapses (see, *e.g.*, Turchin 2003; Nefedov 2004; Malkov, Selunskaja, and Sergeev 2005; Korotayev, Malkov, and Khaltourina 2005). The per capita production to support the above mentioned infrastructures could hardly be lower than 10% of the subsistence level – that is close to Maddison's (2001: 259–60) estimates, which makes it possible to estimate the value of m as \$440, and hence, the value of S_0 as \$4.225.

The correlation between the predicted and observed values for this simulation looks as follows: $R > 0.999$; $R^2 = 0.9986$; $p \ll 0.0001$. For the world population these characteristics are also very high: $R = 0.996$; $R^2 = 0.992$; $p \ll 0.0001$.

According both to our model and the observed data up to the early 1970s we deal with the hyperbolic growth of not only the world population (N), but also per capita surplus production (S) (see Diagram 3.14):

Diagram 3.14. Hyperbolic Growth
of World Per Capita Surplus Production,
in 1990 PPP international dollars
(1 – 1973 CE)

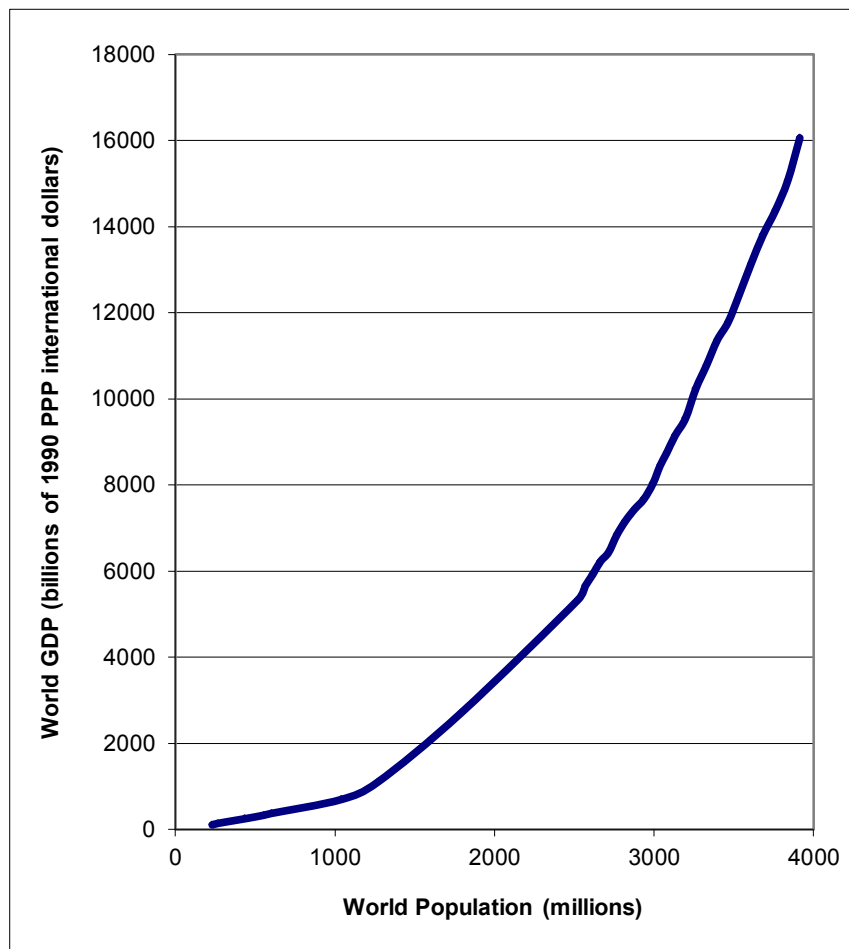


NOTE: Data source – Maddison 2001; Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev (1996: 55–97).

Note that even if S had not been growing, remaining constant, the world GDP would have been growing hyperbolically anyway through the hyperbolic growth of the world population only. However, the hyperbolic growth of S ob-

served during this period of the human history led to the fact that the world population correlated with the world GDP not lineally, but quadratically (see Diagram 3.15):

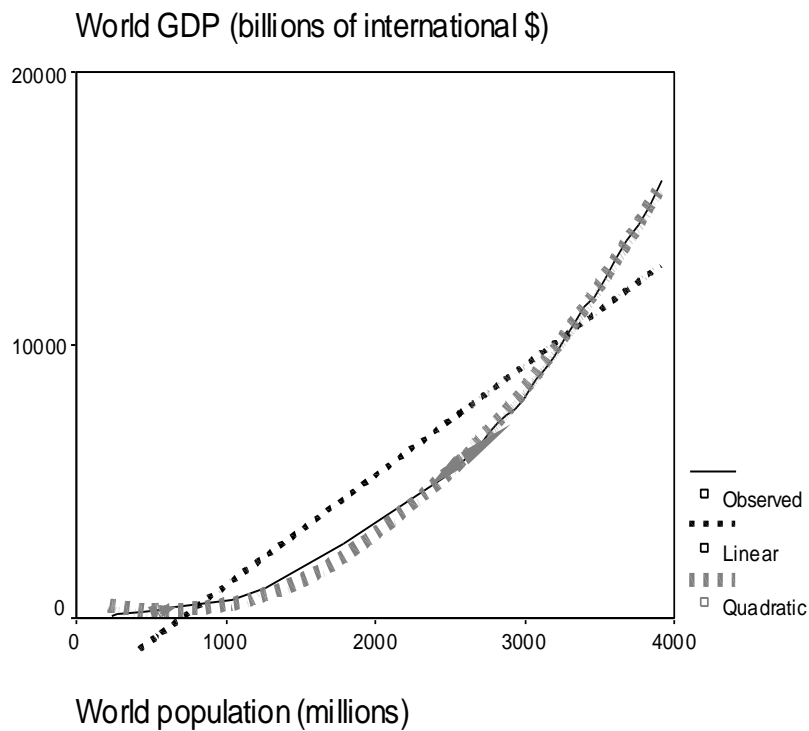
Diagram 3.15. Correlation between World Population and World GDP (1 – 1973 CE)



NOTE: Data source – Maddison 2001; Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev (1996: 55–97).

Indeed, the regression analysis we have performed has shown here an almost perfect ($R^2 = 0.998$) fit just with the quadratic model (see Diagram 3.16):

Diagram 3.16. Correlation between Dynamics of the World Population and GDP Growth (1 – 1973 CE): curve estimations

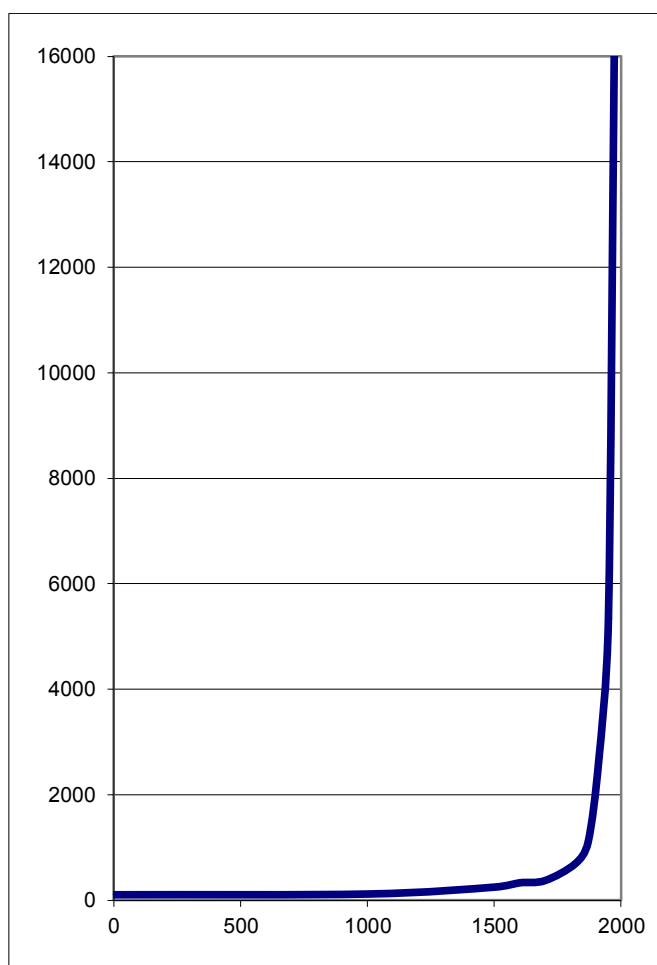


LINEAR REGRESSION: $R^2 = 0.876, p < 0.001$
 QUADRATIC REGRESSION: $R^2 = 0.998, p < 0.001$

As a result the overall dynamics of the world's GDP up to 1973 was not even hyperbolic, but rather quadratic-hyperbolic, leaving far behind the impressive hyperbolic dynamics of the world population growth (see Diagram 3.17):

Diagram 3.17. The World GDP Growth from 1 CE up to the early 1970s

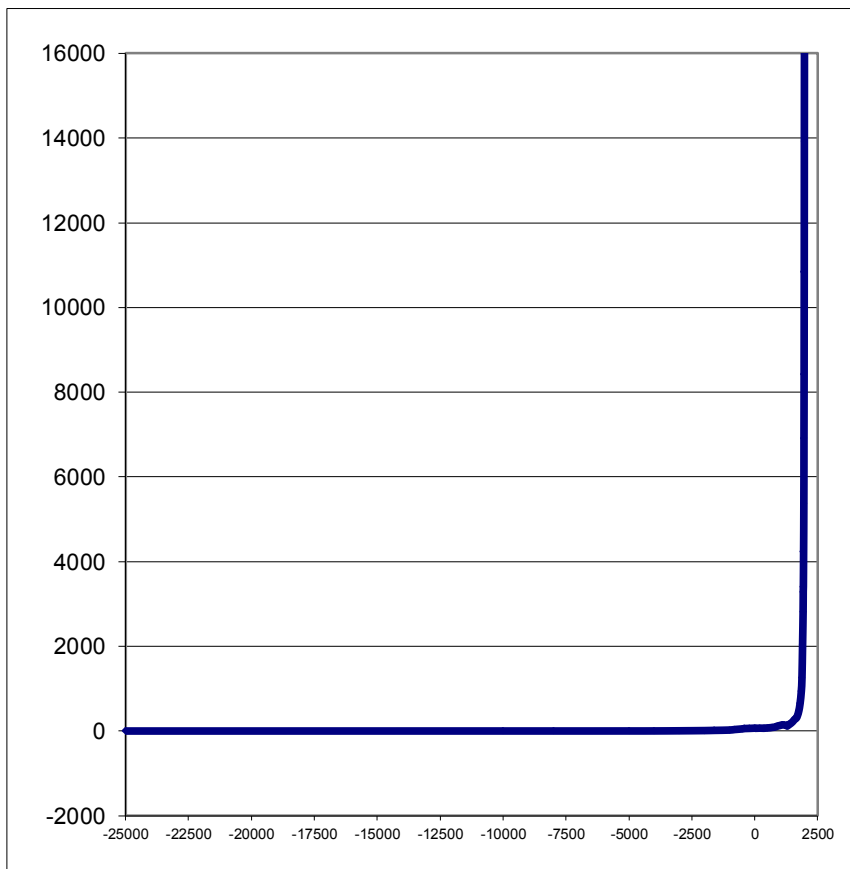
(in billions of 1990 PPP international dollars)



NOTE: Data source – Maddison 2001; Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev (1996: 55–97).

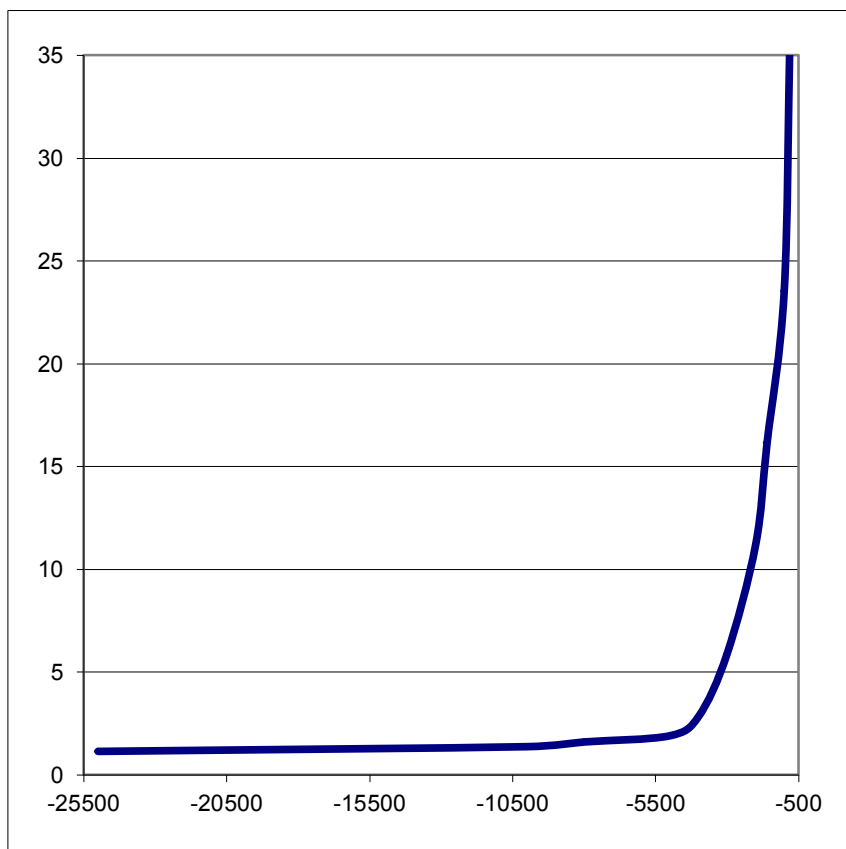
The world GDP growth dynamics would look especially impressive if we take into account the quite plausible estimates of this variable change before 1 CE produced by DeLong (1998), see Diagram 3.18:

Diagram 3.18. World GDP Growth from 25000 BCE up to the early 1970s (in billions of 1990 PPP international dollars)



It is difficult not to admit that in this diagram human economic history appears to be rather "dull", with the pre-Modern era looking as a period of almost complete economic stagnation, followed by explosive modern economic growth. In reality the latter just does not let us discern, in the diagram above, the fact that many stretches of the pre-modern world economic history were characterized by dynamics that was comparatively no less dramatic. For example, as soon as we "zoom in" on the apparently boringly flat stretch of the diagram above up to 800 BCE, we see the following completely dynamic picture (see Diagram 3.19):

Diagram 3.19. World GDP Growth from 25000 BCE up to 800 BCE
(in billions of 1990 PPP international dollars)



This, of course, is accounted for by the difference in scales. During the "iron revolution" the World System economic growth was extremely fast in comparison with all the earlier epochs; however, according to DeLong's estimates in absolute scale it constituted less than \$10 billion a century. At present the world GDP increases by \$10 billion at average every three days (World Bank 2005). As a result, even the period of relatively fast World System economic growth in the age of the "iron revolution" looks like an almost horizontal line in comparison with the stretch of the modern economic growth. In other words the impression of the pre-modern economic stagnation created by Diagram 3.18 could be regarded as an illusion (in the strictest sense of this word) produced just by the quadratic-hyperbolic trend of the world GDP growth up to 1973.

We have already mentioned that, as has been shown by von Foerster, von Hoerner and Kapitza the world population growth before the 1970s is very well approximated by the following equation:

$$N = \frac{C}{t_0 - t}. \quad (2.1)$$

As according to the model under consideration S can be approximated as kN , its long term dynamics can be approximated with the following equation:

$$S = \frac{kC}{t_0 - t}. \quad (3.13)$$

Hence, the long-term dynamics of the most dynamical part of the world GDP, SN , the world surplus product, can be approximated as follows:

$$SN = \frac{kC^2}{(t_0 - t)^2}. \quad (3.14)$$

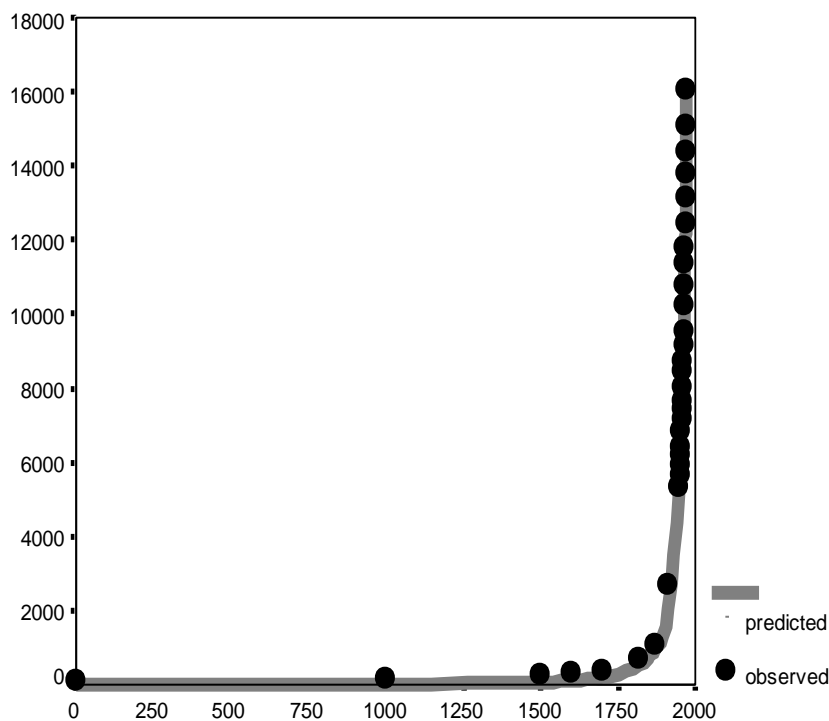
Note that if von Foerster, Mora, and Amiot had had at their disposal, in addition to the world population data, also the data on the world GDP dynamics for 1–1973 (published, however, only in 2001 by Maddison) they could have made another striking "prediction" – that on Saturday, 23 July, 2005 an "economic doomsday" would take place; that is, on that day the world GDP would become infinite. They would have also found that in 1–1973 CE the world GDP growth had followed quadratic-hyperbolic rather than simple hyperbolic pattern.

Indeed, Maddison's estimates of the world GDP dynamics for 1–1973 CE are almost perfectly approximated by the following equation:

$$G = \frac{C}{(t_0 - t)^2}, \quad (3.15)$$

where G is the world GDP, $C = 17355487.3$ and $t_0 = 2005.56$ (see Diagram 3.20):

Diagram 3.20. World GDP dynamics, 1–1973 CE
(in billions of 1990 international dollars, PPP):
the fit between predictions of quadratic-hyperbolic
model and the observed data

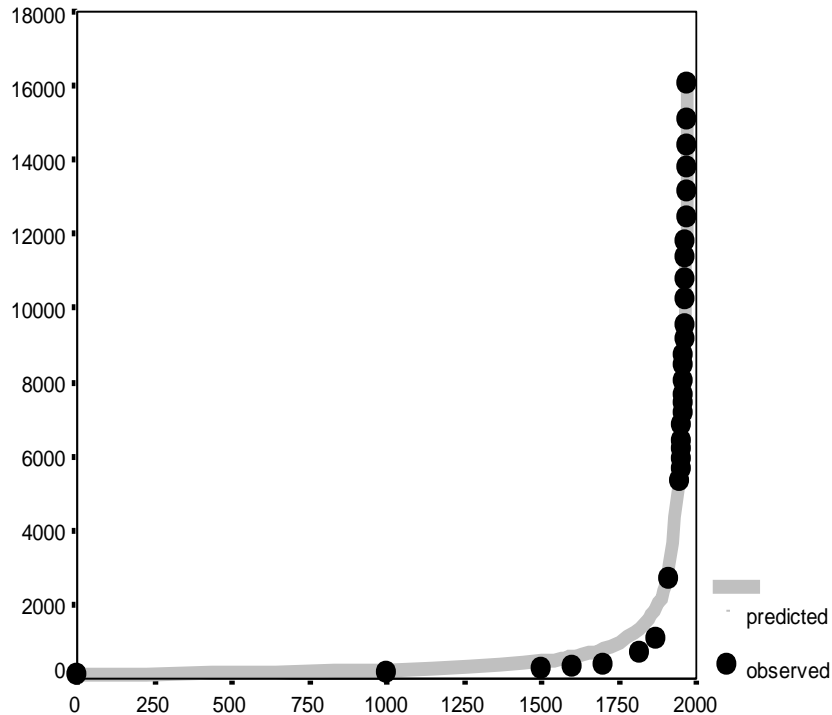


NOTE: $R = 0.9993$, $R^2 = 0.9986$, $p \ll 0.0001$. The black markers correspond to Maddison's (2001) estimates (Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev [1996: 55–97]). The grey solid line has been generated by the following equation: $G = \frac{17749573.1}{(2006 - t)^2}$. The best-fit values of parameters C (17749573.1) and t_0 (2006) have

been calculated with the least squares method (actually, as was mentioned above, the best fit is achieved with $C = 17355487.3$ and $t_0 = 2005.56$ [which gives just the "doomsday Saturday, 23 July, 2005"], but we have decided to keep hereafter to the integer year numbers).

In fact, the fit provided by the simple hyperbolic model for the world GDP dynamics for 1–1973 CE is also in no way bad, but still less perfect than the one found for the quadratic hyperbolic model (see Diagram 3.21):

Diagram 3.21. World GDP dynamics, 1–1973 CE
(in billions of 1990 international dollars, PPP):
the fit between predictions of hyperbolic model
and the observed data



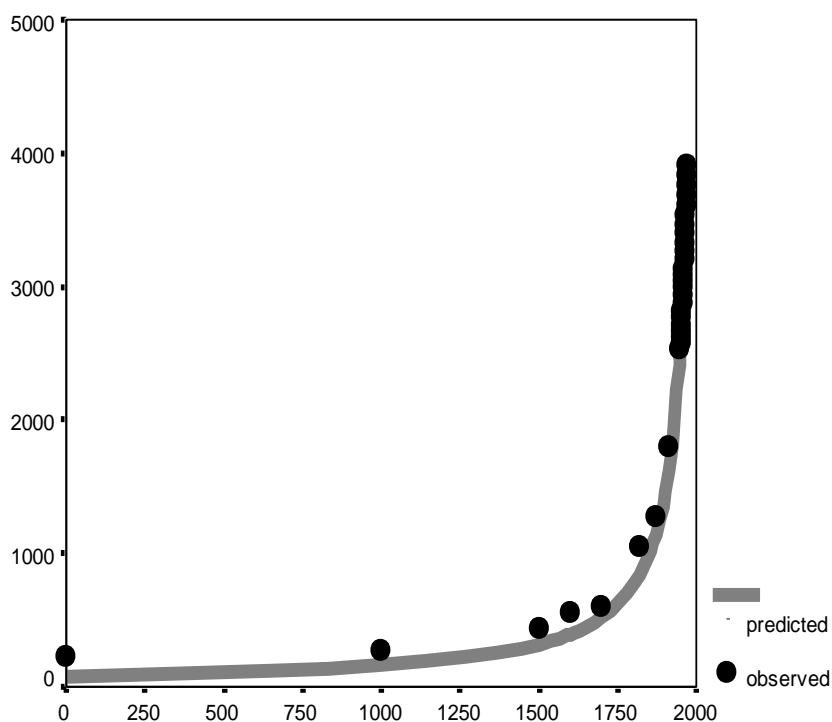
NOTE: $R = 0.9978$, $R^2 = 0.9956$, $p \ll 0.0001$. The black markers correspond to Maddison's (2001) estimates (Maddison's estimates of the world per capita GDP for 1000 CE has been corrected on the basis of Meliantsev [1996: 55–97]). The grey solid line has been generated by the following equation: $G = \frac{227906.1}{1987 - t}$. The best-fit values of parameters C (227906.1) and t_0 (1987) have been calculated with the least squares method.

This, of course, suggests that the long-term dynamics of the world GDP up to the 1970s should be approximated better with a quadratic rather than simple hyperbola.

Note that, on the other hand, the simple hyperbolic model does fit the world population dynamics in 1–1973 much better than the world GDP one. Indeed, though the world 1–1973 CE GDP dynamics fits the simple hyperbolic model of type (2.1) rather well ($R = 0.9978$, $R^2 = 0.9956$, $p \ll 0.0001$), this fit is still

significantly worse than the one observed for the world population growth for the same time ($R = 0.9996$, $R^2 = 0.9991$, $p \ll 0.0001$) (see Diagram 3.22):

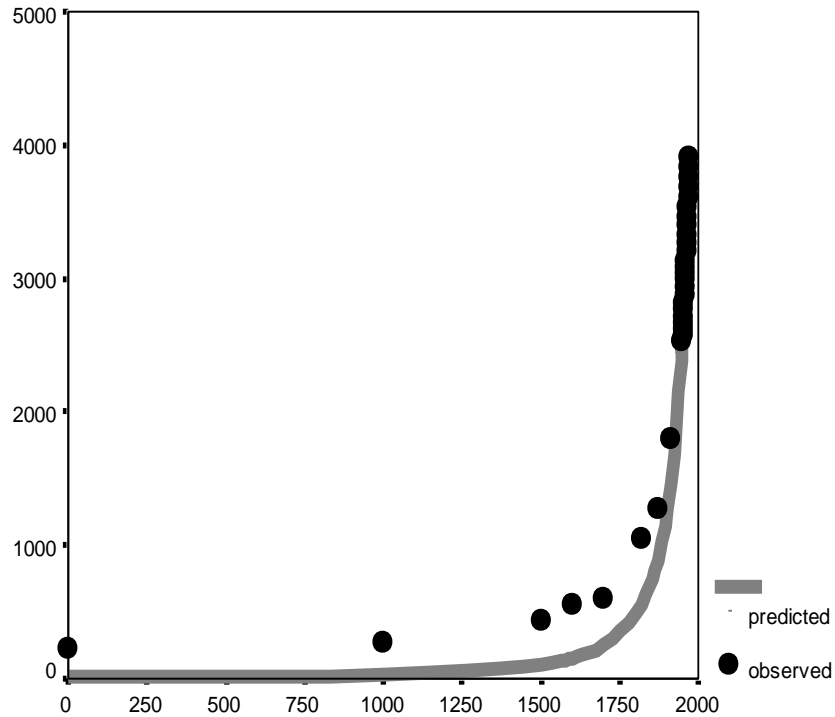
Diagram 3.22. World population dynamics, 1–1973 CE
(in millions): the fit between predictions
of hyperbolic model and the observed data



NOTE: $R = 0.9996$, $R^2 = 0.9991$, $p \ll 0.0001$. The black markers correspond to Maddison's (2001) estimates. The grey solid line has been generated by the following equation: $N = \frac{163158.78}{2014 - t}$. The best-fit values of parameters C (163158.78) and t_0 (2014) have been calculated with the least squares method.

However, the quadratic hyperbolic model renders a worse fit for the world population ($R = 0.9982$, $R^2 = 0.9963$, $p \ll 0.0001$) (see Diagram 3.23):

Diagram 3.23. World population dynamics, 1–1973 CE (in millions): the fit between predictions of quadratic-hyperbolic model and the observed data



NOTE: $R = 0.9982$, $R^2 = 0.9963$, $p \ll 0.0001$. The black markers correspond to Maddison's (2001) estimates. The grey solid line has been generated by the following equation:
$$N = \frac{33505220.7}{(2065 - t)^2}$$

. The best-fit values of parameters C (33505220.702) and t_0 (2065) have been calculated with the least squares method.

Thus, up to the 1970s the hyperbolic growth of the world population was accompanied by the quadratic-hyperbolic growth of the world GDP, just as is suggested by our model.